Phys 410
Spring 2013
Lecture #24 Summary
25 March, 2013

We considered the two-body problem of two objects interacting by means of a conservative central force, with no other external forces acting. The Lagrangian can be simplified by adopting the generalized coordinates: relative coordinate $\vec{r} = \vec{r}_1 - \vec{r}_2$, and the center of mass coordinate $\vec{R} = (m_1\vec{r}_1 + m_2\vec{r}_2)/M$, where $M = m_1 + m_2$ is the total mass. The two-particle Lagrangian simplifies to $\mathcal{L}(\vec{R}, \vec{r}) = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 - U(r)$, where $\mu = m_1 m_2/M$ is called the reduced mass because it is smaller than either m_1 or m_2. Because this Lagrangian is independent of $\vec{R}$, it means that the center of mass (CM) momentum $M\dot{\vec{R}}$ is constant. The other Lagrange equation gives $\mu\ddot{\vec{r}} = -\nabla U(r)$, which is Newton's second law for the relative coordinate.

Taking advantage of the CM conserved momentum, we can jump to the CM (inertial) reference frame, where the CM is at rest, and the two particles are always moving with equal and opposite momenta. In this reference frame, the Lagrangian simplifies to $\mathcal{L} = \frac{1}{2}\mu\dot{\vec{r}}^2 - U(r)$. Because only central forces act, the net torque that the particles exert on each other is zero, hence the total angular momentum of the particles ($\vec{L}$) as seen in this reference frame is conserved. Writing the sum of the angular momenta of the two particles, as seen in the CM reference frame, in terms of the generalized coordinates, we find $\vec{L} = \vec{r} \times \mu\dot{\vec{r}}$, which is the same as the angular momentum of a single particle of mass μ. Because $\vec{L}$ is conserved (including its direction), the vectors $\vec{r}$ and $\dot{\vec{r}}$ must remain in a fixed two-dimensional plane throughout the motion. This means that the motion is strictly two-dimensional!

Now we have to solve the remaining two-dimensional motion problem with this Lagrangian: $\mathcal{L} = \frac{1}{2}\mu\dot{\vec{r}}^2 - U(r)$. Going over to polar coordinates for $\vec{r}$, we get $\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\varphi}^2) - U(r)$. There are two Lagrange equations that follow from this Lagrangian. First we note that φ is an ignorable coordinate, hence the angular momentum of the 'particle' is conserved: $\frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = \mu r^2\dot{\varphi} = constant$. This is in fact just the z-component of the total angular momentum vector $\vec{L}$ that we calculated above. We give it a new name, ℓ, because it is a constant of the motion (you may now recognize the notation from the quantum treatment of the Hydrogen atom). The other Lagrange equation (for r) gives $\mu\ddot{r} = \mu r\dot{\varphi}^2 - dU/dr$. The first term on the RHS is the centripetal acceleration times the mass of the 'particle'. Solving the angular momentum equation for $\dot{\varphi}$ gives $\dot{\varphi} = \ell/\mu r^2$, and the radial equation of motion can be written in terms of ℓ as $\mu\ddot{r} = \ell^2/\mu r^3 - dU/dr$. The first term on the RHS can be written in terms of a derivative as $-\frac{d}{dr}(\ell^2/2\mu r^2)$, so that it can be combined with the potential to create a new "effective potential" $U_{eff}(r) = U(r) + \ell^2/2\mu r^2$. The equation of motion finally reduces to a simple one-dimensional form: $\mu\ddot{r} = -dU_{eff}/dr$.